Matrix Graph: A New Algebraic Graph

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Abstract— In the past Four decades, the study of graph theory has grown beyond leaps and bounds. In one direction, more and more new finer concepts, for example, labelling, dominations and many such graph theoretic phenomenon have been developed. On the other hand, new planar graphs have been constructed using algebraic concepts such as groups, characters and linear transformations of vector spaces. A few examples are the Brattili diagrams [1], the Relative character graph [3], [4], [6], [7].

Keywords: Matrix Graph, Algebraic Graph, Matrix Ring, Lie Algebra, Dominations.

1. INTRODUCTION

This is the first of a series of papers of our attempt to construct yet another new finite, simple, planar graph. (These materials could form a part of the second author's Ph.D thesis, under the supervision of the first author). In this paper, we introduce a new graph called a 'matrix graph' and study some of its properties. Further deep concepts such as connectivity, tree problem, complements, dominations, etc., will be taken up in the subsequent papers.

2. Basic Concepts from Matrix Theory

We assume that all the matrix entries are complex numbers.

2.1 Definition: Let A be an $n \times n$ matrix. A complex number λ is an eigen value of A if λ is a root of the characteristic polynomial $|A - \lambda I|$
(I be an $n \times n$ identity matrix). The following basic facts may be recalled.

(i) A has n eigen values, not necessarily distinct.
(ii) A is invertible if and only if 0 is not an eigen value.
(iii) If A is invertible and $\lambda_1, \lambda_2 \dots \lambda_n$ are its eigen values, then ${}^1/_{\lambda_1}, {}^1/_{\lambda_2}, \dots, {}^1/_{\lambda_n}$ are the eigen values of A^{-1} and $\lambda_1^r, \lambda_2^r, \dots, \lambda_n^r$ are the eigen values of A^r for any positive integer r.
(iv) A satisfies the expansion of the polynomial expression $|A - \lambda I|$ in λ (Cayley-Hamilton theorem).

It is important to note that even if the entries of A are reals, the eigen values need not be real. For example, consider the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

The Characteristic polynomial is

$$\det\left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} = \lambda^2 + 1.$$

The roots are i, -i.

3. The Matrix Graph

Let $S = \{A_1, A_2, \dots, A_p\}$ be a collection of $n \times n$ matrices over $\mathbb{C}$. We define a graph $\Gamma_s = (V, E)$ where the vertex set V is the p matrices in S and two distinct matrices A_i and A_j of S are adjacent if and only if, A_i and A_j share a common eigen value.

If S is fixed, we sometimes write Γ for Γ_s, clearly Γ is a finite, simple, undirected planar graph.

3.1 Proposition: Let $S = \{A_1, A_2, \dots, A_p\}$ where each A_i is either an upper or lower triangular matrix, with diagonal entries$(d_{i1}, d_{i2}, \dots d_{in})$ with $d_{ik} \neq d_{jl}$ for $i \neq j$. Then Γ_s is the null graph.

Proof: Clearly the eigen values of A_i are $d_{i1}, d_{i2}, \dots d_{in}$ and those of A_j are $d_{j1}, d_{j2}, \dots d_{jn}$. By assumption $d_{ik} \neq d_{jl}, i \neq j$.

After deleting the $d_s{}'$ with the same second entries, we see that the remaining entries corresponding to A_i and $A_j (i \neq j)$ are all distinct. Hence A_i and A_j have no common eigen value, which means that the corresponding graph Γ_s has no edges.

The corresponding discussion for complete graphs will be taken up a little later. We shall first go for a criterion for adjacency of A_i and A_j.

But first we recall the following well-known result, whose proof is easy, but not trivial.

3.2 Proposition: Let A and B be two $n \times n$ matrices having a common eigen value α. Let $P_A(\lambda)$ and $P_B(\lambda)$ be the characteristic polynomials of A and B respectively. Then

$$\det P_A(B) = P_B(A) = 0.$$

Proof: Write $P_A(\lambda) = (\lambda - \alpha_1)(\lambda - \alpha_2) \dots (\lambda - \alpha_n)$ where $\alpha_1, \alpha_2 \dots \alpha_n$ are the eigen values of A. Let $\alpha_1 = \alpha$. Substituting B for λ in the above, we get, $P_A(B) = (B - \alpha_1 I)(B - \alpha_2 I) \dots (B - \alpha_n I)$.

Then

$det P_A(B) = det(B - \alpha_1 I).det(B - \alpha_2 I)\dots. det(B - \alpha_n I)$.

Since $\alpha = \alpha_i$ is an eigen value of B as well, $det(B - \alpha_1 I) = 0$. Hence $det_A(B) = 0$. Arguing similarly inter-changing A_i and A_j we get $det_B(A) = 0$.

3.3 Corollary: A_i and $A_j$are adjacent in the previous notation, if and only if $det_{Ai}(A_j) = det_{Aj}(A_i) = 0$.

3.4. Proposition: Let $\{b_1, b_2, \dots, b_p\}$ be distinct positive integers. Define $n \times n$ matrices $S = \{A_{\sigma_1}, A_{\sigma_2}, \dots, A_{\sigma_p}\}$ as follows:

Take $X = \{1, 2, \dots p\}$ and let $\sigma_1, \sigma_2, \dots \sigma_p$ be distinct permutations of X.

Put $A_{\sigma_i} = diag(\sigma_i(1), \sigma_i(2), \dots \sigma_i(p))$. Then the graph Γ_s is complete.

Proof: $A_{\sigma_1}, A_{\sigma_2}, \dots, A_{\sigma_p}$ are all distinct $p \times p$ matrices, but all of them have the same set of eigen values $\{\sigma_i(1), \sigma_i(2), \dots \sigma_i(p)\}$. Hence Γ_s is clearly complete.

A few more basic results stated below will be useful. For details we refer to [5].

I. Any $A \in M_n(\mathbb{C})$ (the full matrix ring of $n \times n$ matrices over $\mathbb{C}$) is similar to an upper triangular matrix.

II. A and B are unitarily similar if there exists a unitary matrix P (ie., $\bar{P}P^T = I$) such that $P^{-1}AP = B$. Then any $A \in M_n(\mathbb{C})$ is unitarily similar to an upper triangular matrix.

III. If rank A = 1, then trace A (= sum of diagonal entries) is an eigen value of A.

Since similar matrices of the same order have the same eigen values, our construction of Γ revolves around (similarity of) matrices similar to certain known forms.

3.5 Proposition: Let $S = \{A_1, A_2, \dots, A_p\}$ where rank $A_i = 1$ for each i. Then in Γ_s, A_i is adjacent to A_j if and only if trace A_i = trace A_j

Proof: From III, A_i and A_j have one common eigen value (=trace). Hence they are adjacent,

3.6 Example:

A typical example from Lie Algebras.

The simplest Lie Algebra is the usual $L = sl_2(\mathbb{C})$ the Lie algebra of matrices of trace 0.

Take, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then A, B and C form a basis of L (as a vector space).

We can take S = {A, B}. (rank A = rank B = 1). Then clearly the graph Γ_s is

Next we can take $L = sl_3(\mathbb{C})$ of dimension 8 and look for a corresponding construction.

In fact, we can generalize this to the following.

3.7 Proposition. The set S of all submatrices of the basic matrices of the Lie algebra $L = sl_{l+1}^{l}(\mathbb{C})$ of rank l forms a complete matrix graph.

Proof: Imitating the steps in 3.6 we see that the elements of S clearly satisfy all the required conditions: rank1 and trace 0. Hence any two vertices of S are adjacent proving the completeness of Γ_s.

We can go to matrices of the group $gl_n(\mathbb{C})$ (where trace 0 restriction is not there).

It may turn out that the resulting graphs are not complete.

3.8 Corollary: Let $S = \{A_1, A_2, \dots, A_p\}$ where each A_i is $E_{mn}(m \neq n)$ an elementary matrix then Γ_s is complete.

Proof: Trivial

3.9 Proposition:

Let $S = \{A_1, A_2, \dots, A_p\}$ where each A_i is a $n \times n$ permutation matrix. Then Γ_s is complete.

Proof: Each A_i has 1 at only one row and only one column and all other entries are 0's. If X denotes the column vector $(1,1,\dots,1)^T$ then it is easily seen that $A_i(1,1,\dots,1)^T = (1,1,\dots,1)^T$ for all i; ie., $A_iX = X$ showing that 1 is an eigen value of each A_i and hence Γ_s is complete.

3.10 Remark 1: Monomial matrices are those having a non zero entry in a unique column and row. Our question is whether the above proposition goes through for monomial matrices also. For simplicity first take $n = p = 2$.

Let $A = \begin{pmatrix} 0 & a_1 \\ a_2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$

$|A - \lambda I| = \lambda^2 - a_1a_2$, giving $\lambda = \pm\sqrt{a_1a_2}$.

$|B - \lambda I| = \lambda^2 - \lambda(b_1 + b_2) + b_1b_2$, which gives $\lambda = (b_1, b_2)$. Required adjancency condition is

$b_1 = \pm\sqrt{a_1a_2}\, , b_2 = \mp\sqrt{a_1a_2}$.

Remark 2: The graph Γ_s can be complete without any of the above conditions. For instance, the matrices A_1, A_2, A_3 given below do not have any of the conditions said above.

$A_1 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix},$

$A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{pmatrix}.$

The eigen values of A_1, A_2 and A_3 are the following triplets in the same order:

(1, 1, 3), (2, 2, 3) and (1, 2, 4). Obviously Γ is complete.

It seems that the question of characterizing the completeness of Γ_s, for a given S, is not easy! In this context we have the following interesting algorithm for completeness.

4. An Algorithm

4.1 Let $S = \{A_1, A_2, \dots, A_p\}$ be a set of $n \times n$ symmetric matrices. Diagonalize each A_i to get the corresponding diagonal matrix

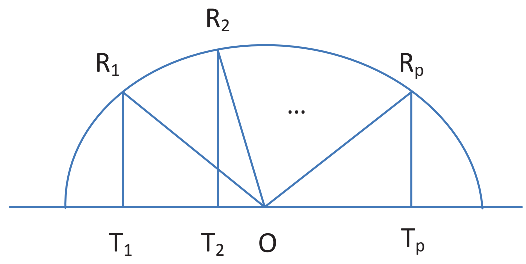
$D_i = diag(\alpha_{i1}, \alpha_{i2}, \dots \alpha_{in})$, $i = 1,2,\dots,p$ so that α_{ik}'s are precisely the eigen values of D_i for every i. Then,

I. If $\alpha_{ik} = \alpha_{jl}$, for every pair i, j and for some k, l then Γ_s is complete.

II. If for any given A_i there exists α_{ik} such that

$\alpha_{ik} = \alpha_{jl}, i \neq j$ such that for some k and l then Γ_s is a connected graph.

We shall discuss the above in a very special case.



Consider the matrices $S = \{A_1, A_2, \dots, A_p\}$ given by $A_i = \begin{pmatrix} a_i & c_i \\ c_i & -a_i \end{pmatrix}$ where $a_i = |OT_i|$ and $C_i = |R_iT_j|$ in the above semi-circle of radius r. Then clearly, for each i,

$a_1^2 + c_1^2 = a_2^2 + c_2^2 = \dots = a_p^2 + c_p^2 = r^2$.

Hence the eigen values of A_i are $\pm r$.This proves that the graph Γ_s is complete.

Another interesting variation is the following. Continuing our earlier notation,

III. If a_i's and c_i's are chosen so that

$a_i^2 + c_i^2 \neq a_j^2 + c_j^2$ for any pair (i,j) then it is easily seen that the corresponding graph is the null graph. For a proof, one can easily verify that the eigen values of A_i are$\pm\sqrt{a_i^2 + c_i^2}$.

5. Unitarily similar matrices

First recall that an $n \times n$ matrix A over $\mathbb{C}$ is unitarily similar to an upper triangular matrix T. This means that there exists a unitary $n \times n$ matrix P (ie., $P\bar{P}^T = I$) and an upper triangular matrix T such that satisfying $\bar{P}^T AP = T$. Here

(P and) T need not be unique but A and T do have the same eigen values.

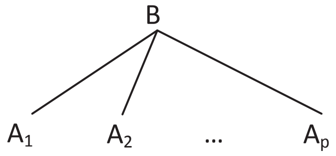
5.1 Definition: Let X be the collection of all matrices T satisfying the above conditions. Let S be a finite subset of X. We then get the corresponding matrix graph Γ_s. We denote this by $\Gamma_{s,A}$ and call this the **unitary matrix graph** of A. Once again Γ_s is complete.

We obtain an interesting result to get a tree.

5.2 Proposition: Let $S = \{B,\ A_1,\ A_2, \dots A_p\}$ be $n \times n$ matrices such that

i) no two A_i's have the same eigen value.
ii) B has 1 as an eigen value
iii) $A_iB = BA_i$ for all i. Then Γ_s is a tree, in fact a star.

Proof: By the choice of eigen values $\{A_i\}$, no two $A_i's$ are adjacent. By a basic result from matrix theory (see [5]) eigen values of A_iB are products of eigen value of A_i and B Since 1 is an eigen value of B, if a_i is an eigen value of A_i, then a_i is an eigen value of A_iB too. Hence B and A_i are adjacent for each i. Hence Γ_s is a tree, a star.



6. Some applications.

6.1. Definition: An $n \times n$ matrix $A = (a_{ij})$ is **stochastic** if $a_{ij} \geq 0$ for all i, j and $\sum_{i=1}^{n} a_j$ for all j. It is an interesting fact that if A is stochastic, then 1 is an eigen value of A. We have the following easy

6.2 Proposition: If S is a set of p stochastic $n \times n$ matrices, Γ_s is complete.

Proof: Each A in S has 1 as an eigen value and hence any two matrices in S are adjacent.
In this way we can associate probability theory with graph theory.

6.3 Liapunov Matrix Graphs.
If A, B, C are $n \times n$ matrices over $\mathbb{C}$, an equation of the form $AX + XB = C$ is known as the **Sylvester matrix equation**.

6.4 Proposition. The Sylvester matrix equation $AX + XB = C$ has a unique solution if and only if $Spec\ A \cap -Spec\ B = \varphi$. (Spec A denotes the spectrum of A, ie, the set of all eigen values of A). For a proof we refer to [5].

A special case is the **Liapunov matrix equation** $\bar{A}^T X + XA = -I$. (*)
(It is used in a stability criterion for the linear differential equation $\frac{dt}{dx} = Ax$.) We say that the matrix A is **stable** if all its eigen values lie in the left hand half plane.

A well known result in stability theory says that A is stable if and only if there exists a positive definite solution X to the equation $\bar{A}^T X + XA = -I$. It follows that A is stable if and only if there exists a positive definite (hermitian) matrix X which is a solution of $\bar{A}^T X + XA = -I$.

6.5 Definition: Let S be a finite set of stable matrices. The **stability matrix graph** is defined as the graph Γ_s of matrix graph obtained by the (unique) solution matrices X_A for each A in S. Note that condition (*) is automatically satisfied by A and $\bar{A}^T$.

The advantage of the matrix X_A is that it is hermitian and positive definite. When n and p are large, computation part will be easier.

6.6. The Rayleigh Matrix Graph.
Let A be an $n \times n$ hermitian matrix. Let v be a nonzero vector in $\mathbb{R}^n$. The Reyleigh quotient for A at the vector v is the real number $\rho(v) = \frac{\bar{v}^T A v}{\bar{v}^T v}$.
(Since A is Hermitian, $\rho(v)$ must be real.) If A is real symmetric then $\rho(v) = \frac{v^T A v}{v^T v}$.

6.7 Theorem: Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of an $n \times n$ hermitian (or real symmetric) matrix A written so that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

Then for each
$v \neq 0, \lambda_1 \leq \rho(v) \leq \lambda_n$. It also follows that if v_i is an eigen vector for λ_i then $\rho(v_i) = \lambda_i$ for each i.

In particular $\lambda_n = \rho(v_n) = Max\{v, v \neq 0\}$ and $\lambda_1 = \rho(v_1) = Min\{v, v \neq 0\}$ .The **Rayleigh matrix graph** $\Gamma_{s,A}$, is defined as follows:
Let A be a hermitian (a real symmetric) matrix such that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Choose nonzero vectors, $v_1, v_2, \ldots, v_p$ such that
$\rho(v_1) < \rho(v_2) < \cdots < \rho(v_p)$. Then $\rho(v_i) = \lambda_i$
for each i (by the above theorem).
Let $\| v \| = (x_1^2 + \cdots + x_n^2)^{1/2}$
$(v = (x_1, x_2, \ldots, x_n))$ be the Euclidean norm on $\mathbb{R}^n$.

Let $S = (w_1, w_2, \ldots, w_p)$ be 'p' n-vectors such that $\| w_1 \| < \| w_2 \| < \cdots < \| w_p \|$. The vectors in S form the vertices of our new graph called the Rayleigh matrix graph of A relative to S and with two vectors w_r and w_s are adjacent $(r < s)$ if and only if there exist a pair (i, j) such that $\lambda_i \leq \| w_r \|$ and $\| w_s \| \leq \lambda_j$. This graph could be a tool in approximation and perturbation theory.

7.Neighbourhood Matrix Graph.

We now generalize our matrix graph using little bit of analysis.

7.1 Take S as before. Then Γ_s absolute graph is the graph having adjacency if and only if A_i and A_j have at least one common value in absolute value.

7.2.Definition: Let $\delta \geq 0$ be a real number. Let $S = \{A_1, A_2, \ldots, A_p\}$ be 'p' $n \times n$ complex matrices. Then the δ-neighbourhood matrix graph $\Gamma_{\delta,S}$ of S relative to δ is the graph (V,E) where V, as before denotes the matrices $\{A_i\}$ and two A_i, A_j $(i \neq j)$ are adjacent if and only if $A_i$has one eigen value α_i and $A_j$has one eigen value α_j such that $|\alpha_i - \alpha_j| \leq \delta$.This definitely generalizes absolute matrix graphs by taking $\delta = 0$.

Absolute graph and neighbourhood graphs vastly generalized and lead us into real time application situations. First, due to matlab techniques,we can actually locate eigen values of a given (real) $n \times n$ matrix A . Then one can approximate eigen values of A by means of another matrix B all whose entries are 'close' enough to those of A in a neighbourhood sense, ie., by a famous 'continuity theorem' [5] which says that if entries of A and the corresponding entries of B are 'arbitrarily close', then the eigen values of A and B are 'sufficiently close'. Finally, coupling this with the 'Greshgorin discs' $D_i\left(= Z \in \mathbb{C}/|z - a_{ii}| \leq \sum_{\substack{i=1 \\ j\neq 1}}^{n} |a_{ij}|\right)$ one can further reach deeper results.

CONCLUSION

In conclusion, we say that the construction of the matrix graph of a finite collection of matrices of the same order is new and we hope that we can continue to analyse further deep graph theoretic aspects such as connectivity, domination, etc., of this graph. Also neighbourhood matrix graphs could vastly be seen in the application context, using matlab techniques, including QR-algorithms.

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